the arc differs very slightly from the horizontal distance, or $ds=dx$ nearly; and the following is the resulting equation:—

$$y=x\tan\phi+\mathrm{A}\left\{\frac{n}{2(n+1)}a^{\frac{2(n+1)}{n}}+a^{\frac{n+2}{n}}x-\frac{n}{2(n+1)}(x+a)^{\frac{2(n+1)}{n}}\right\},$$

where $\mathrm{A}=\frac{g}{\mathrm{C}^{\frac{2}{n}}}\cdot\frac{n}{n+2}$, and c and a are the constants, and n the index in the general equation

$$(x+a)\,v^n=\mathrm{C}.$$

Examples of the application of this are given, showing the calculated elevation for the 12-pounder muzzle-loading Armstrong gun for ranges of 2855 yards and 4719 yards, the gun being 17 feet above the planes.

The calculated elevations were 6° 56′ and 14° 6′, the actual elevations being 7° and 15° respectively.

It is not intended to claim more than approximate accuracy for the formulæ in this paper. The general formula has been shown to be derived by taking mean values of n and c, whereas the actual results would indicate that the value of n increases with the diameter of the projectile; and it is shown in a note that the values of n which agree best with experiment are,

for the small shot $n=2{\cdot}4$,

for the large shot $n=4$,

corresponding to the following resistances,

small shot $\mathrm{R}=v^{4{\cdot}4}$,

large shot $\mathrm{R}=v^6$.

Whether in reality the index does increase with the diameter of the shot must be left to be determined by more extended experiments; meantime it may be assumed that the general formula in this paper represents with tolerable accuracy the law of resistance and the loss of velocity of projectiles varying from 8·8 lbs. to 251 lbs. in weight, from 3 inches to 9 inches in diameter, and from 1500 to 600 feet per second in velocity.

II. "On the Theory of Probability, applied to Random Straight Lines." By M. W. Crofton, B.A., of the Royal Military Academy, Woolwich, late Professor of Natural Philosophy in the Queen's University, Ireland. Communicated by Prof. Sylvester. Received February 5, 1868.

(Abstract.)

This paper relates to the Theory of Local Probability—that is, the application of Probability to geometrical magnitude. This inquiry seems to have been originated by the great naturalist Buffon, in a celebrated problem proposed and solved by him. Though the subject has been more than once touched upon by Laplace, yet the remarkable depth and beauty of this new Calculus seem to have been little suspected till within the last

few years, when the attention of several English mathematicians has been directed to it, and results of a most singular character have been obtained.

The problems on Local Probability which have been hitherto treated relate almost exclusively to *points* taken at random. The object of the present paper is to show how the Theory of Probability is to be applied to *straight lines* whose position is unknown, or, in other words, which are taken at random.

The author commences by showing that when a straight line is drawn at random in an indefinite plane, or, in other words, when we take one out of an infinite assemblage of lines all drawn at random in the plane, the true mathematical conception of this assemblage is as follows:—

Conceive the plane ruled with an infinity of parallels at a constant infinitesimal distance (δp) asunder; then imagine this system of parallels turned through an infinitesimal angle ($\delta\theta$); then through a second equal angle, and so on, till the parallels return to their original direction; the plane will thus be covered with an infinite number of systems of parallels, running in every possible direction.

If an infinite plane be covered in this manner with straight lines, and we draw any closed convex contour on the plane, and then imagine all the lines effaced from the plane, except those which meet this contour, we shall have a clear conception of the system of random lines which meet the given contour.

By applying mathematical calculation to this system, the following important principle is proved:—

The measure of the number of random lines which meet a given closed convex contour is L, *the length of the contour.*

If the contour be non-convex, or be not closed, the measure will be *the length of an endless string passing round it and tightly enveloping it.*

Hence, given any closed convex contour of length L, and any other of length l, lying wholly within the former, the probability that a line drawn at random to meet L shall also meet l, is

$$P=\frac{l}{L}.$$

The following propositions are then established:—

If the contour l lie wholly *outside* L, then, if X be the length of an endless band tightly enveloping the two contours and crossing between them, and Y the length of another endless band also enveloping both, but not crossing between them, the probability that a random line meeting L shall also meet l, is

$$p=\frac{X-Y}{L}.$$

Again, if the contour l should intersect L (whether in two or more points), then, if Y be an endless band tightly enveloping both,

$$p=\frac{L+l-Y}{L}.$$

A closed convex boundary of any form, of length L, encloses an area Ω: if two random straight lines intersect it, the probability of their intersection lying within it is

$$p=\frac{2\pi\Omega}{L^2}.$$

The probability of their intersection lying within any given area, ω, which is enclosed within Ω, is

$$p=\frac{2\pi\omega}{L^2}.$$

A more difficult question would be to determine the probability in the case where ω is external to Ω.

These fundamental results, it will be observed, are of great generality. The author proceeds to apply them to the solution of various problems relating to random straight lines; in fact any such problem of probability may be reduced by the principles above laid down to a question of pure mathematical calculation.

What will probably be considered among the most curious results contained in this paper are the collateral applications of the theory to the integral calculus. Several integrals of a singular character are obtained, some of which it seems very difficult to prove by any known method. One or two of these are subjoined, with indications of the methods used in establishing their truth.

If a given convex boundary be intersected by a system of random lines, as above described, every pair of lines will meet in a point; and these points of intersection will be scattered all over the plane, some within the boundary, some without. Those within will evidently be distributed with uniform density over the area; but it becomes a question for those outside, to determine the law according to which their density varies; and it is proved in this paper that *the density of the intersections of a system of random lines crossing a given area, for any external point* P, *is proportional to* $\theta-\sin\theta$, *where* θ *is the apparent angular magnitude of the area from* P.

Hence the number of external intersections is represented by

$$\iint(\theta-\sin\theta)\,dS;$$

now the number of internal will be $\pi\Omega$, and the whole number $\frac{1}{2}L^2$. Hence

If Ω *be any plane area, enclosed by a convex boundary of length* L, *and* θ *the angle which it subtends at any external point* (x, y), then

$$\iint(\theta-\sin\theta)\,dxdy=\tfrac{1}{2}L^2-\pi\Omega,$$

*the integral extending over the whole external surface of the plane**.

By conceiving an infinite system of random lines covering an infinite

* This theorem has appeared in the 'Comptes Rendus' of the French Academy of Sciences (Dec. 1867).

plane, and a second system, all of which meet a given boundary in that plane, and then fixing our attention on the intersections of the former system with the latter, we find the density here proportional to θ; and the following theorem is deduced from this consideration:—

Given any convex boundary (whose apparent magnitude is called θ), let there be an external boundary surrounding it, such that any tangent to the inner cuts off a constant area from the outer, then

$$\iint \theta dx dy = \pi \mathrm{D},$$

the integral extending over the whole annulus between them, D *being the difference of the areas of the parts into which the annulus is divided by any tangent to the inner.*

For instance, we may take two similar concentric ellipses. If both the inner and outer boundaries are of any convex forms whatever, the above expression is still true, provided D mean *the average value of the difference of areas as the tangent revolves by uniform angular displacements.*

If we consider a plane covered with random lines, and then divide them into two systems, one crossing a given boundary, the other all outside it, the density of the points in which the former system cut the latter will be proportional to $\sin\theta$; and this leads to the next theorem.

If an endless string (of length Y) *be passed round a given convex boundary (of length* L), *and the string be kept stretched by the point of a pencil, which thus traces out an external boundary, then if θ be the apparent magnitude of the given boundary at any point* (x, y), we shall have

$$\iint \sin \theta dx dy = \mathrm{L}\,(\mathrm{Y}-\mathrm{L}),$$

the integral extending over the annular space between the boundaries.

A remarkable instance of this is an ellipse, the outer curve being, as is well known, a confocal ellipse.

Some other applications of the theory to integration are then given. It is important to notice that these applications, though having arisen from researches on probability, rest on a basis wholly independent of that theory. The apparatus of equidistant parallels revolving by infinitesimal angular displacements, which has been here employed, is a purely geometrical conception; and the proofs of these integrals can be presented in a strict mathematical form. A reticulation composed of two systems of parallels crossing at a finite angle has already been employed by Cauchy, Liouville, and Eisenstein as a method in the theory of numbers and elliptic functions. The reticulation used above is a more delicate and complicated one, consisting, not of two, but of an infinite number of systems of parallels.

There remains a more difficult but deeply interesting inquiry, scarcely touched upon in this paper—namely, the extension of the above results to the cases of straight lines, and of planes, taken at random in space.